Talk

1) Intro

2) Cayley graphs for finitely presented groups

   i) Notation

   ii) Van Kampen diagram

   iii) Lemma

3) Borel asymptotic dimension

   i) Corollary 5.5
   ↳ Every Borel action of poly. growth. group has finite Borel asym. d.

   ii)

1) Intro

- Joint with Jenna and Matt

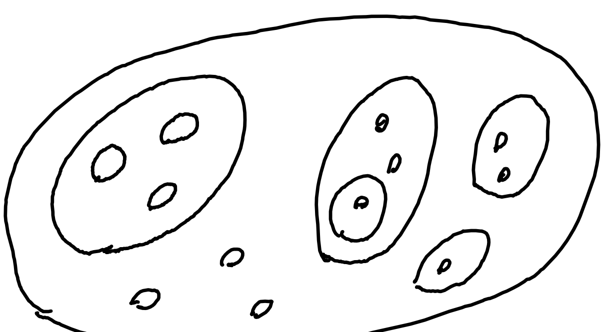
Goal: Construct connected toasts in free Borel actions of polynomial growth, one-ended groups.

Defn If E is a CBER with graphing G, a collection $\Delta \subseteq X^{<\mathbb{N}}$ of finite subsets is a connected toast if:

- $\forall A \in \Delta$, $A - \bigcup_{\substack{B \in \Delta \\ B \subsetneq A}} B$ is connected.

- $\forall A, B \in \Delta$, $B_1(A) \cap B = \emptyset$, $B_1(A) \subseteq B$ or $B_1(B) \subseteq A$.

- $\forall e = (x,y) \in G$, $\exists A \in \Delta$, $x, y \in A$.



- Joint with Jenna and Matt

2] Balloons & von Kampen diagrams

- Context
  - Γ a group with finite presentation $\Gamma = \langle S, R \rangle$ (S symmetric)
  - $G := Cay(\Gamma)$ the Cayley complex w.r.t. to S, R.

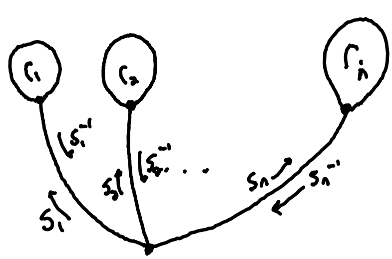
- Balloons

Let w be a word of S trivial in Γ.

i.e. $w \in \langle\langle R \rangle\rangle$

$$w = \prod_{i=1}^{n} s_i r_i s_i^{-1} \qquad s_i \text{ words in } S, \quad r_i \in R.$$

w corresponds to some "bundle of ballons"



This "bundle" is simply connected, planar.

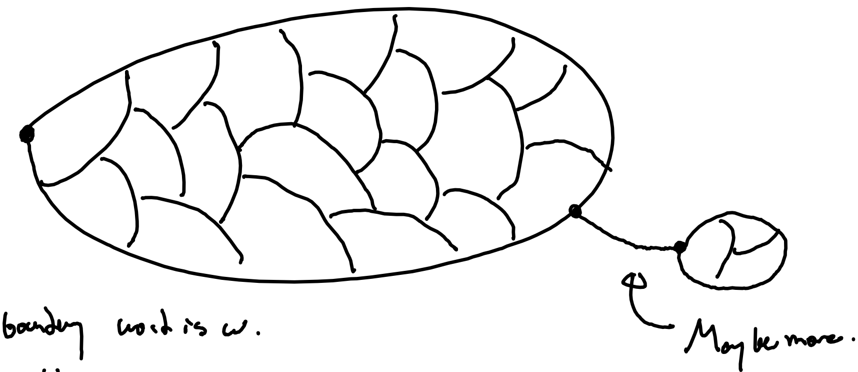
Suppose $s_1 = \tilde{s} \cdot s_1'$, $s_2 = \tilde{s} \cdot s_2'$.

We can glue the strings.

If $s_2 = s_1 \cdot s_2'$,

Further, suppose $r_1 = r_1' \cdot \tilde{s}$, $s_2' = r_1' \cdot s_2''$

After doing this fully, we reach the van Kampen diagram associated to w.



The boundary word is w.

⚠ Embeddability

<u>vK Lemma</u>

To every trivial word in Γ there is a vK diagram with it as a boundary.

To any vK diagram, the boundary word is trivial.

- Uniform boundary connectedness.

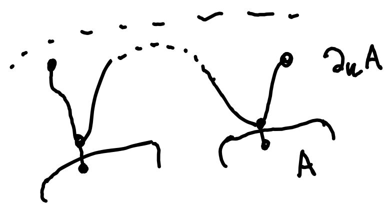
<u>Proposition</u>

Let $K := \max_{r \in R} \{|r|\} + 1$! Let $A \subseteq \Gamma$ s.t.

- A^c is connected
- $B_K(A) := \{ \gamma \in \Gamma : d(\gamma, A) \leq K \}$ is connected

Then $\partial_K(A) := \{ \gamma \in \Gamma : 1 \leq d(\gamma, A) \leq K \}$ is connected.

Proof It suffices to show that for any $x, y \in \partial_1 A$, there is a path between x, y contained in $\partial_r A$.

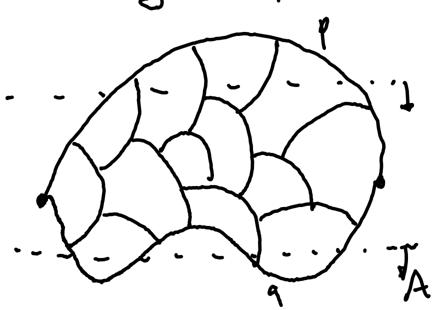


Let x, y be as above.

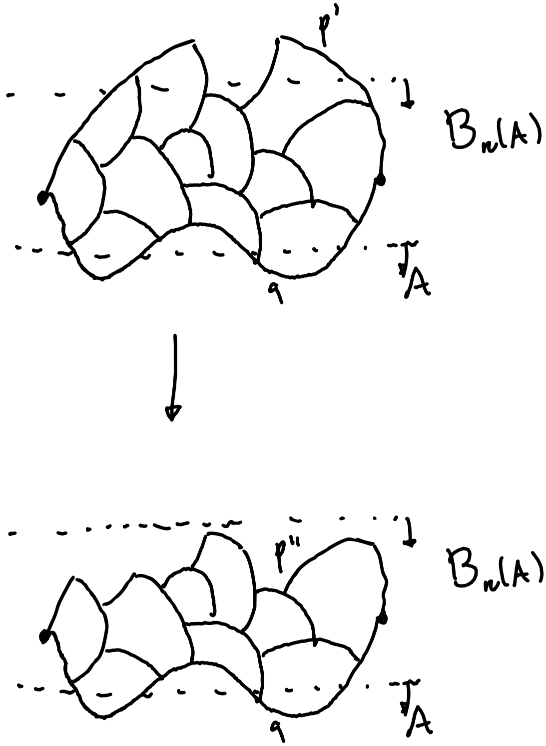
Pick paths p from x to y in A^c

q from x to y in $B_r(A)$

Pick vK diag of pq^{-1}



We can remove generating relators whose boundary crosses $B_r(A)^c$



Thus, in the end, we are left with some path p'' contained in $B_n(A)$. Further, p'' does not pass through A. If it did, some relators $r \in R$ would have vertices in $B_n(A)^c$ and A, contradicting that $K > |r|$. ■

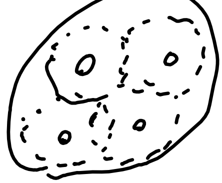
<u>Consequence</u> If a potential connected tract Δ satisfy

i) $\forall A, B \in \Delta$, $B_n(A) \cap B = \emptyset$

$B_n(A) \subseteq B$ or $B_n(B) \subseteq A$

we get ii) $\forall A \in \Delta$, $A - \bigcup_{\substack{B \in \Delta \\ B \subsetneq A}} B$ is connected for free.

Picture to have in mind:



3) Borel Asymptotic Dimension.

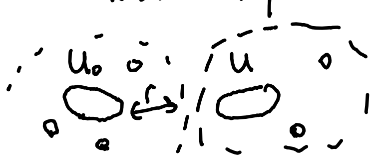
- Context Γ as before

$\Gamma \curvearrowright X$ free Borel action, G Shreier graph

$\operatorname{asdim}_B(\Gamma \curvearrowright X) = d < \infty$

Definition For all $r>0$, there are $\mathcal{U}_0, \dots, \mathcal{U}_d$ Borel cover of X s.t. $\mathcal{F}_r(\mathcal{U}_i)$ is uniformly bounded eq. rel.

where $x \, \mathcal{F}_r(\mathcal{U}_i) \, y \Leftarrow x, y \in \mathcal{U}_i, \ d(x,y) \leq r$



Prop If r_n is an increasing sequence, there are Borel covers $\mathcal{U}_0^n \dots \mathcal{U}_d^n$ s.t.

- $\mathcal{F}_{r_n}(\mathcal{U}_i^n)$ is uniformly bounded (for each n)
- If $n<m$, $x \in \mathcal{U}_i^n$, either $B_{r_n}(x) \subseteq \mathcal{U}_i^m$ or $\cap \, \mathcal{U}_i^m = \emptyset$

Prop

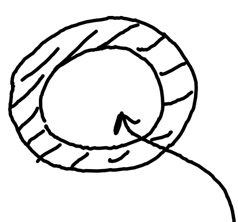
For $\Gamma \curvearrowright X$ a free Borel action of one-ended, polynomial growth group $\langle S \mid R \rangle$

Let $K := \max_{r \in R} \{|r|\} + 1$, $r >> K$

There are V_n s.t.

- $\partial_r(V_n)$ is uniformly bounded for each n.
- $\forall x \in X$, $\exists n$ s.t. $B_r(x) \subseteq V_n$
- If $n < m$, $x \in V_n \to B_{4r}(x) \subseteq V_m$
  $\cap V_m = \emptyset$

Problem: These may be ringlike



Solution $\to$ complete them by adding finite components of V_n^c.

You preserve uniform boundedness.

preserve $B_{4r}(x)$ condition.